

# A Note on the Voting Problem

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## Abstract

Let  $v(n)$  be the minimum number of voters with transitive preferences which are needed to generate any strong preference pattern (ties not allowed) on  $n$  candidates. Let  $k = \lfloor \log_2 n \rfloor$ . We show that  $v(n) \leq n - k$  if  $n$  and  $k$  have different parity, and  $v(n) \leq n - k + 1$  otherwise.

## 1 Introduction

Let us consider a set of  $n$  candidates or options  $A = \{a, b, c, \dots\}$  which are ordered by order of preference by each individual of a set  $U$  of voters. Thus, each  $\alpha \in U$  can be identified with a permutation  $\alpha = x_1 x_2 \cdots x_n$  of the elements of  $A$ , where  $x_i$  is preferred over  $x_j$  (denoted  $x_i \rightarrow x_j$ ) if and only if  $i < j$ . The set of voters determine what is called a *preference pattern* which summarizes the majority opinion about each pair of options.

In this note only *strong* preference patterns are considered, that is, it is assumed that there are no ties. So, each preference pattern on  $n$  options is fully represented by a tournament  $T_n$  on  $n$  vertices where the arc  $(a, b)$  means  $a \rightarrow b$ , that is,  $a$  is preferred over  $b$  by a majority of voters. Conversely, given any pattern  $T_n$  we may be interested in finding a minimum set of voters, denoted  $U(T_n)$ , which generates  $T_n$ . Let  $v(T_n) = |U(T_n)|$  and let  $v(n) = \max\{v(T_n)\}$  computed over all tournaments with  $n$  vertices. In [2] McGarvey showed that  $v(n)$  is well defined, that is, for any  $T_n$  there always exist a set  $U(T_n)$  and  $v(n) \leq 2\binom{n}{2}$ . Sterns [3] showed that  $v(n) \leq n + 2$  if  $n$  is even and  $v(n) \leq n + 1$  if  $n$  is odd. Finally, Erdős and Moser [1] were able to prove that  $v(n)$  is of the order  $O(n/\log_2 n)$ . In fact all the above results were given for preference patterns which are not necessarily strong (in this case a tie between  $a$  and  $b$  can be represented either by an absence of arcs between  $a$  and  $b$  or by an edge  $\{a, b\}$ ). It is worth noting that, contrarily to the method of Erdős and Moser, the approaches of McGarvey and Sterns give explicit constructions of a set of voters which generate any desired pattern. In the case of strong patterns we improve the results of the latter authors by giving an inductive method to obtain a suitable set of voters.

## 2 Strong preference patterns

Let us begin with a very simple but useful result, which is a direct consequence of the fact that in our preference patterns there are no ties.

**Lemma 2.1.** *Let  $v(n)$  be defined as above. Then,  $v(n)$  is odd.*

**Proof.** By contradiction, suppose that, for a given strong pattern  $T_n$ ,  $v(n)$  is even. Then, for any two options  $a, b$  we have that either  $a \rightarrow b$  or  $b \rightarrow a$  with at least two votes of difference. Consequently, the removing of a voter does not change the preference pattern.  $\square$

Notice that, from this lemma, Sterns' result particularized for strong patterns are  $v(n) \leq n + 1$  for  $n$  even and  $v(n) \leq n$  for  $n$  odd.

Our results are based on the following theorem.

**Theorem 2.2.** *Let  $T_{n+2}$  be a strong pattern containing two options, say  $a$  and  $b$ . Let  $T_n = T_{n+2} \setminus \{a, b\}$ . Then,  $v(T_{n+2}) \leq v(T_n) + 2$ .*

**Proof.** Let  $U(T_n) = \{\alpha_1, \alpha_2, \dots, \alpha_r\}$  be a minimum set of  $r = v(T_n)$  voters generating  $T_n$ . By Lemma 2.1,  $r$  is odd. Besides, suppose without loss of generality that  $a \rightarrow b$ , and consider the sets  $A_1 = \{x \neq a \mid x \rightarrow b\}$  and  $A_2 = \{x \neq b \mid a \rightarrow x\}$ . Assuming  $A_1 \cap A_2 \neq \emptyset$ ,  $A_1, A_2$  (any other case follows trivially from this one), we can write  $A_1 = \{y_1, y_2, \dots, y_s, \dots, y_t\}$  and  $A_2 = \{y_s, y_{s+1}, \dots, y_t, \dots, y_m\}$ ,  $1 < s \leq t < m$ . Now, let us define the sequences  $\gamma = y_1 y_2 \dots y_{s-1}$ ,  $\delta = y_s y_{s+1} \dots y_t$ ,  $\sigma = y_{t+1} y_{t+2} \dots y_m$  and  $\mu = y_{m+1} y_{m+2} \dots y_n$ , and consider the following set of  $r + 2$  voters:

$$\begin{aligned} \beta_i &= b\alpha_i a, & 1 \leq i \leq (r+1)/2, \\ \beta_j &= a\alpha_j b, & (r+3)/2 \leq j \leq r, \\ \beta_{r+1} &= \gamma a \delta b \sigma \mu, \\ \beta_{r+2} &= \bar{\mu} a \bar{\sigma} \bar{\delta} \bar{\gamma} b, \end{aligned}$$

where  $\bar{\gamma} = y_{s-1} \dots y_2 y_1$ ,  $\bar{\delta} = y_t \dots y_{s+1} y_s$ , etc. Now it is routine to verify that these voters generate the pattern  $T_{n+2}$  and, hence,  $v(T_{n+2}) \leq r + 2 = v(T_n) + 2$ .  $\square$

A tournament or strong preference pattern  $T$  is called *transitive* if  $a \rightarrow b$  and  $b \rightarrow c$  implies  $a \rightarrow c$ . In this case it is clear that  $v(T) = 1$ . The proof of the following result can be found in [1].

**Theorem 2.3** ([1]). *Let  $f(n)$  be the maximum number such that every tournament on  $n$  vertices has a transitive subtournament on  $f(n)$  vertices. Then,*

$$\lfloor \log_2 n \rfloor + 1 \leq f(n) \leq 2 \lfloor \log_2 n \rfloor + 1.$$

The proof of the lower bound, due to Sterns, gives a very simple algorithm to find a subtournament which attains such a bound, see again [1].

From Theorems 2.2 and 2.3 we get the following corollary.

**Corollary 2.4.** *Given  $n \geq 2$ , set  $k = \lfloor \log_2 n \rfloor$ . Then  $v(n) \leq n - k$  if  $n$  and  $k$  have different parity, and  $v(n) \leq n - k + 1$  otherwise.*

**Proof.** Let  $T_n$  be any tournament on  $n$  vertices. First, use Theorem 2.3 to find a transitive subtournament  $T$  on  $k + 1$  vertices. If  $n$  and  $k$  have different parity, then  $n - k - 1$  is even. So, starting from  $T$ , we can apply Theorem 2.2 repeatedly,  $(n - k - 1)/2$  times, to obtain a set of  $n - k$  voters which generates  $T_n$ . Otherwise, we consider a subtournament of  $T$  on  $k$  vertices and proceed as above with the remaining  $n - k$  vertices.

□

## References

- [1] P. Erdős and L. Moser, On the representation of directed graphs as unions of orderings, *Publ. Math. Inst. Hung. Acad. Sci.* **9** (1964), 125–132.
- [2] D.C. McGarvey, A theorem on the construction of voting paradoxes, *Econometria* **21** (1953), 608–610.
- [3] R. Stearns, The voting problem, *Amer. Math. Monthly* **66** (1959), 761–763.